4. Let x, y, and z be positive real numbers such that $x + y + z = 1$. For a positive integer n, let $S_n = x^n + y^n + z^n$. Also, let $P = S_2 S_{2005}$ and $Q = S_3 S_{2004}$.

(a) Find the smallest possible value of Q.

(b) If x, y, and z are distinct, determine which of P or Q is the larger.

Solution by Arkady Alt, San Jose, CA, USA.

(a) By the Power Mean inequality, for any positive integer k we have

$$\begin{aligned} \frac{x^k + y^k + z^k}{3} &\geq \left(\frac{x+y+z}{3}\right)^k = \frac{1}{3^k}, \\ S_k \geq x^k + y^k + z^k &\geq \frac{1}{3^{k-1}} \geq \frac{1}{3^{k-1}}, \end{aligned}$$

where equality occurs if and only if $x = y = z = \frac{1}{3}$. Thus the minimum value of Q is $\frac{1}{3^2} \cdot \frac{1}{3^{2003}} = \frac{1}{3^{2005}}$.

(b) We will prove that if x, y, z are distinct, then

$$\frac{S_{n+1}}{S_n} > \frac{S_n}{S_{n-1}}$$

holds for any positive integer n. Indeed,

$$\begin{aligned} & S_{n+1}S_{n-1} - S_n^2 \\ &= (x^{n+1} + y^{n+1} + z^{n+1})(x^{n-1} + y^{n-1} + z^{n-1}) - (x^n + y^n + z^n)^2 \\ &= x^{n+1}(y^{n-1} + z^{n-1}) + y^{n+1}(z^{n-1} + x^{n-1}) + z^{n+1}(x^{n-1} + y^{n-1}) \\ &\quad -2(x^n y^n + y^n z^n + z^n x^n) \\ &= \sum_{\text{cyclic}} (x^{n+1}y^{n-1} + x^{n-1}y^{n+1} - 2x^n y^n) \\ &= \sum_{\text{cyclic}} x^{n-1}y^{n-1}(x-y)^2 \geq 0\,. \end{aligned}$$

Therefore, $\frac{S_{n+1}}{S_n} > \frac{S_m}{S_{m-1}}$ for $n \geq m \geq 2$, or $S_{m-1}S_{n+1} > S_m S_n$. In particular for $m = 3$, $n = 2004$ we have $S_2 S_{2005} > S_3 S_{2004}$.

Next are solutions to the Hong Kong Team Selection Test 2, given at [2009 : 214–215].

1. Let $ABCD$ be a cyclic quadrilateral. Show that the orthocentres of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, and $\triangle DAB$ are the vertices of a quadrilateral